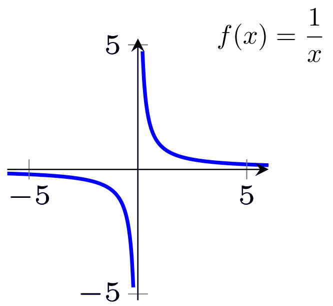
Objectives:

- Find limits where the variable goes to infinity and/or the limit is infinite.
- Find vertical and horizontal asymptotes of a given function.
- Choose and use an appropriate strategy to use with a given indeterminant form.

Limits Involving Infinity Graphically:

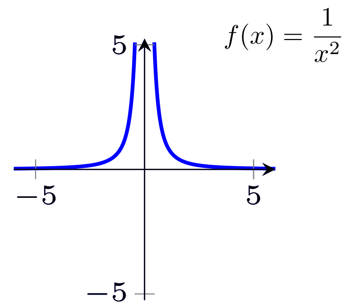
- $\lim_{x\to a^+} f(x) = \pm\infty$ and $\lim_{x\to a^-} f(x) = \pm\infty$ represent vertical asymptotes.
- $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to -\infty} f(x) = L$ represent horizontal asymptotes.

Common Functions with Limits Involving Infinity:



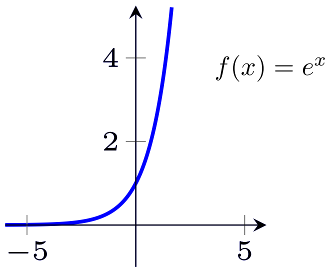
$\lim_{x\to\infty} \frac{1}{x} = 0$ $\quad$ $\lim_{x\to 0^+} \frac{1}{x}$ " $=$ " ∞

$\lim_{x\to -\infty} \frac{1}{x} = 0$ $\quad$ $\lim_{x\to 0^-} \frac{1}{x}$ " $=$ " $-\infty$

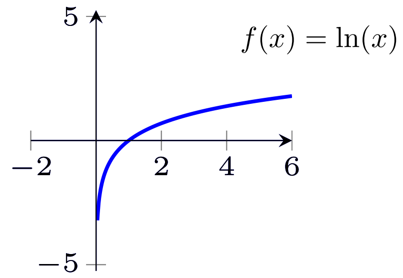


$\lim_{x\to\infty} \frac{1}{x^2} = 0$ $\quad$ $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$

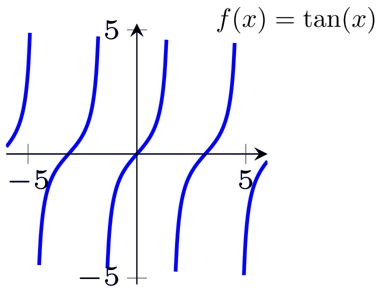
$\lim_{x\to -\infty} \frac{1}{x^2} = 0$ $\quad$ $\lim_{x\to 0^-} \frac{1}{x^2} = \infty$



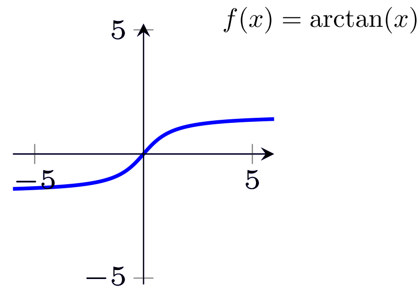
$\lim_{x\to\infty} e^x = \infty$ $\quad$ $\lim_{x\to -\infty} e^x = 0$



$\lim_{x\to\infty} \ln(x) = \infty$ $\quad$ $\lim_{x\to 0^-} \ln(x) = -\infty$



$\lim_{x\to\frac{\pi}{2}^-} \tan(x) = \infty$ $\lim_{x\to\frac{\pi}{2}^+} \tan(x) = -\infty$



$\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$ $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$

Using These Common Functions:
Be careful with composite functions! Remember that the direction of the limit of the outside function depends on whether the inside function is increasing or decreasing.

1. $\lim_{x\to\infty} e^{\frac{1}{x}} = \lim_{a\to0^+} e^a = 1$

2. $\lim_{x\to0^+} e^{\frac{1}{x}} = \lim_{a\to\infty} e^a = \infty$

3. $\lim_{x\to0^+} \ln(2^x) = \lim_{a\to1^+} \ln(a) = 0$

4. $\lim_{x\to\infty} \frac{1}{\ln x} = \lim_{a\to\infty} \frac{1}{a} = 0$

5. $\lim_{x\to\infty} \ln\left(\frac{1}{x}\right) = \lim_{a\to\infty} \frac{1}{a} = \lim_{a\to0^+} \ln(a) = -\infty$

6. $\lim_{x\to\infty} \sin(\arctan x) = \lim_{a\to\pi/2} \sin(a) = \sin\left(\frac{\pi}{2}\right) = 1$

Indeterminate Forms
Remember that we call the form "$\frac{0}{0}$" indeterminate. The forms $\frac{\infty}{\infty}$ and $\infty - \infty$ are also indeterminate. (We will see even more types of indeterminate forms later on.)

Useful Strategy:
If $\lim_{x\to\infty} f(x)$ is of the form "$\frac{\infty}{\infty}$", try multiplying the numerator and denominator by 1 over the highest power of x in the denominator.

Indeterminate Form Examples:

1. $\lim_{x\to\infty} \dfrac{2x^2+3}{x^2+x}$ (of the form "$\frac{\infty}{\infty}$")

$$\lim_{x\to\infty} \frac{2x^2+3}{x^2+x} = \lim_{x\to\infty} \frac{(2x^2+3)\frac{1}{x^2}}{(x^2+x)\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2+\frac{3}{x^2}}{1+\frac{1}{x}} = \frac{2}{1} = 2$$

So y=2 is a horizontal asymptote.

2. $\lim_{x\to\infty} \dfrac{3x-1}{x^2+4}$ (of the form "$\frac{\infty}{\infty}$")

$$\lim_{x\to\infty} \frac{3x-1}{x^2+4} = \lim_{x\to\infty} \frac{(3x-1)\frac{1}{x^2}}{(x^2+4)\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{3}{x}-\frac{1}{x^2}}{1+\frac{4}{x^2}} = \frac{0}{1} = 0$$

So y=0 is a horizontal asymptote.

3. Find horizontal asymptotes of $f(x) = \dfrac{5x^2+7}{2x-4}$ (both limits below are of the form "$\frac{\infty}{\infty}$")

$$\lim_{x\to\infty} \frac{(5x^2+7)\frac{1}{x}}{(2x-4)\frac{1}{x}} = \lim_{x\to\infty} \frac{5x+\frac{7}{x}}{2-\frac{4}{x}} = \infty$$

$$\lim_{x\to-\infty} \frac{(5x^2+7)\frac{1}{x}}{(2x-4)\frac{1}{x}} = \lim_{x\to-\infty} \frac{5x+\frac{7}{x}}{2-\frac{4}{x}} = -\infty$$

So there are no horizontal asymptotes.

4. $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$ (of the form "$\frac{\infty}{\infty}$")

$$\lim_{x\to\infty} \frac{x\left(\frac{1}{x}\right)}{\sqrt{x^2+1}\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}\sqrt{\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1$$

5. $\lim_{x\to\infty} (\sqrt{x^2+1}-x)$ (of the form "$\infty-\infty$" which we can't compute)

$$\lim_{x\to\infty} (\sqrt{x^2+1}-x) = \lim_{x\to\infty} \frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{(\sqrt{x^2+1}+x)} = \lim_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0$$

6. $\lim_{x\to-\infty} (\sqrt{x^2+1}-x)$ Not indeterminant; this just looks like "$\infty+\infty$" so the limit is infinite.